

# Fluctuation dynamo at finite correlation times using renewing flows

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Fluctuation dynamos are generic to turbulent astrophysical systems. The only analytical model of the fluctuation dynamo, due to Kazantsev, assumes the velocity to be delta-correlated in time. This assumption breaks down for any realistic turbulent flow. We generalize the analytic model of fluctuation dynamo to include the effects of a finite correlation time,  $\tau$ , using renewing flows. The generalized evolution equation for the longitudinal correlation function  $M_L$  leads to the standard Kazantsev equation in the  $\tau \rightarrow 0$  limit, and extends it to the next order in  $\tau$ . We find that this evolution equation involves also third and fourth spatial derivatives of  $M_L$ , indicating that the evolution for finite  $\tau$  will be non-local in general. In the perturbative case of small- $\tau$  (or small Strouhal number), it can be recast using the Landau-Lifschitz approach, to one with at most second derivatives of  $M_L$ . Using both a scaling solution and the WKBJ approximation, we show that the dynamo growth rate is reduced when the correlation time is finite. Interestingly, to leading order in  $\tau$ , we show that the magnetic power spectrum, preserves the Kazantsev form,  $M(k) \propto k^{3/2}$ , in the large  $k$  limit, independent of  $\tau$ .

## 1. Introduction

The continued existence of magnetic fields in most astrophysical systems is thought to be due to dynamo action which converts kinetic energy of the plasma into magnetic energy. In particular, fluctuation dynamos are generic, and operate with minimal requirements of the underlying fluid flow. A random flow with modest magnetic Reynolds number  $R_M \sim 100$  is sufficient to activate the fluctuation dynamo. Here  $R_M = u/(q\eta)$  with  $u$  and  $q$  respectively characteristic velocity and wavenumber of the flow and  $\eta$  is the resistivity. Hence fluctuation dynamos are considered to be ubiquitous in all astrophysical plasmas.

The analytical theory for the fluctuation dynamo was given by Kazantsev (1967). A dynamical equation for the two point magnetic correlator was derived by using a simple model for the velocity field which is delta-correlated in time. This assumption of delta-correlation allows one to convert the stochastic induction equation for the magnetic field to a partial differential equation in real space for the longitudinal magnetic correlation function  $M_L(r, t)$ . Its solution clearly showed for the first time that a random flow with modest  $R_M$  can lead to the growth of the field. Kazantsev then also predicted that the magnetic power spectrum for a single scale or a large  $P_M$  turbulent flow, scales asymptotically as  $M(k) \propto k^{3/2}$ , for  $q \ll k \ll k_\eta$ , with  $k_\eta$ , the wavenumber where resistive dissipation becomes important. This spectrum is known as the Kazantsev spectrum.

Following the seminal work of Kazantsev (1967), there has been considerable interest in fluctuation dynamos, in terms of theoretical developments, in terms of their direct

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simulation and in terms of various astrophysical applications (Molchanov *et al.* 1985; Zeldovich *et al.* 1990; Kulsrud & Anderson 1992; Subramanian 1997; Rogachevskii & Kleeorin 1997; Subramanian 1999; Chertkov *et al.* 1999; Haugen *et al.* 2004; Schekochihin *et al.* 2004, 2005; Brandenburg & Subramanian 2005; Subramanian *et al.* 2006; Enßlin & Vogt 2006; Cho *et al.* 2009; Malyskin & Boldyrev 2010; Federrath *et al.* 2011; Tobias *et al.* 2011; Sur *et al.* 2012; Schober *et al.* 2012; Beresnyak 2012; Brandenburg *et al.* 2012; Bhat & Subramanian 2013). These works have clearly demonstrated that random (or turbulent) flows in a conducting plasma, with  $R_M > R_{\text{crit}} \sim 30 - 500$ , leads to the amplification of magnetic fields on the fast eddy turn over time scale, usually much smaller than the age of the astrophysical system. The  $R_{\text{crit}}$  depends on  $P_M = \nu/\eta$ , where  $\nu$  is the viscosity and could even depend on the forcing wavenumber (Subramanian & Brandenburg 2014). This rapid growth implies that fluctuation dynamos are crucial for the early generation of magnetic fields in primordial stars, galaxies and galaxy-clusters. It is therefore important to obtain a clear understanding of the fluctuation dynamo.

Note that the feature of delta-correlation in time, assumed by Kazantsev (1967), is not realistic in turbulent astrophysical plasmas. There the correlation time,  $\tau$ , is expected to be at least of the order of the smallest eddy turn over time. Thus, its important to understand the effects of finite time correlation on the fluctuation dynamo. This is the main motivation of the present work.

The effect of having a finite- $\tau$  on the magnetic energy growth has been considered by Chandran (1997), while Schekochihin & Kulsrud (2001) examined its consequences for the single point PDF in the ideal limit. The correction to the evolution of the two point correlator due to having a finite- $\tau$  was considered by Kleeorin *et al.* (2002); they however seem to have kept only a subset of the terms we derive here. It was shown by Mason *et al.* (2011) that the results from simulations involving finite- $\tau$  velocity flows, can be matched to the predictions using the Kazantsev equation provided the diffusivity spectrum is appropriately filtered out at small-scales. An analytic understanding of the magnetic spectrum at finite- $\tau$  is however still lacking.

The present work uses random flows with finite time correlation known as renewing (or renovating) flows to develop an analytic generalization of the results of Kazantsev (1967) to include the effects of a finite correlation time. Zeldovich *et al.* (1988) had used renewing flows for studying the diffusion of scalars and the generation of vectors in random flows. Such flows have also been used to study the effect of finite correlation time on mean field dynamos (Dittrich *et al.* 1984; Gilbert & Bayly 1992; Kolekar *et al.* 2012). In an earlier letter (Bhat & Subramanian 2014) (hereafter BS14), we gave a brief account of the our work on fluctuation dynamos using renewing flows, emphasizing an intriguing result that the Kazantsev spectrum is in fact preserved even for such finite- $\tau$ . In the present paper, we present our detailed derivations of the generalized Kazantsev equation and the results in BS14, as well as some new WKB analysis. In the next section, we formulate the basic problem of fluctuation dynamos in renewing flows. The detailed derivation of the evolution equation for  $M_L(r, t)$  which incorporates finite- $\tau$  effects, to the leading order is given in section 3. Scaling and WKB analysis of this generalized evolution equation is taken up in section 4, and we end with a discussion of our results.

## 2. Fluctuation dynamo in renewing flows

The evolution of magnetic field, in a conducting fluid with velocity  $\mathbf{u}$ , is given by the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}). \quad (2.1)$$

The velocity field is a random flow which renews itself every time interval  $\tau$  (Dittrich *et al.* 1984; Gilbert & Bayly 1992) and was given by Gilbert & Bayly (1992)(GB) as,

$$\mathbf{u}(\mathbf{x}) = \mathbf{a} \sin(\mathbf{q} \cdot \mathbf{x} + \psi), \quad (2.2)$$

with  $\mathbf{a} \cdot \mathbf{q} = 0$  for an incompressible flow. In each time interval  $[(n-1)\tau, n\tau]$ ,

- (i)  $\psi$  is chosen uniformly random between 0 to  $2\pi$ ;
- (ii)  $\mathbf{q}$  is uniformly distributed on a sphere of radius  $q = |\mathbf{q}|$ ;
- (iii) for every fixed  $\hat{\mathbf{q}} = \mathbf{q}/q$ , the direction of  $\mathbf{a}$  is uniformly distributed in the plane perpendicular to  $\mathbf{q}$ .

Specifically, for computational ease, we modify the GB ensemble and use,

$$a_i = \tilde{P}_{ij} A_j, \quad \tilde{P}_{ij}(\hat{\mathbf{q}}) = \delta_{ij} - \hat{q}_i \hat{q}_j \quad (2.3)$$

where  $\mathbf{A}$  is uniformly distributed on a sphere of radius  $A$ , and projects  $\mathbf{A}$  to the plane perpendicular to  $\mathbf{q}$ . Then on averaging over  $a_i$  and using the fact that  $\mathbf{A}$  is independent of  $\mathbf{q}$ , we have  $\langle u \rangle = 0$  and,

$$\begin{aligned} \langle a_i a_l \rangle &= \langle a^2 \rangle \frac{\delta_{il}}{3} = \left\langle A_j A_k \tilde{P}_{ij} \tilde{P}_{lk} \right\rangle = A^2 \frac{\delta_{jk}}{3} \left\langle \tilde{P}_{ij} \tilde{P}_{lk} \right\rangle = \frac{A^2}{3} \left\langle \tilde{P}_{il} \right\rangle = \frac{2A^2}{3} \frac{\delta_{il}}{3} \\ &\Rightarrow \langle a^2 \rangle = 2A^2/3 \end{aligned} \quad (2.4)$$

This modification in ensemble does not affect any result using the renewing flows. Condition (i) on  $\psi$  ensures statistical homogeneity, while (ii) and (iii) ensure statistical isotropy of the flow.

The magnetic field evolution in any time interval  $[(n-1)\tau, n\tau]$  is

$$B_i(\mathbf{x}, n\tau) = \int \mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0) B_j(\mathbf{x}_0, (n-1)\tau) d^3 \mathbf{x}_0 \quad (2.5)$$

where  $\mathcal{G}_{ij}(\mathbf{x}, \mathbf{x}_0)$  is the Green's function of Eq. (2.1). S: added below We define the magnetic two-point spatial correlation function as

$$\langle B_j(\mathbf{x}, t) B_l(\mathbf{y}, t) \rangle = M_{jl}(r, t), \quad \text{where } r = |\mathbf{r}| = |\mathbf{x} - \mathbf{y}|, \quad (2.6)$$

and  $\langle . \rangle$  denotes an ensemble average. Here we have assumed the statistical homogeneity and isotropy of the magnetic field. Note that if the initial field is statistically homogeneous and isotropic, then this is preserved by the renewing flow that we consider as we show explicitly below. Then the evolution of the fluctuating field defined by the two point correlation is,

$$M_{ih}(|\mathbf{x} - \mathbf{y}|, n\tau) = \int \left\langle \tilde{\mathcal{G}}_{ijhl}(\mathbf{x}, \mathbf{x}_0, \mathbf{y}, \mathbf{y}_0, \tau) \right\rangle M_{jl}(|\mathbf{x}_0 - \mathbf{y}_0|, (n-1)\tau) d^3 \mathbf{x}_0 d^3 \mathbf{y}_0. \quad (2.7)$$

where  $\langle . \rangle$  around  $\tilde{\mathcal{G}}$  denotes the average over the ensemble described above. Here we could split the averaging on the right side of equation between the Greens function and the initial magnetic correlator, because the renewing nature of flow implies that the Greens function in the current interval is uncorrelated to the magnetic correlator from the previous interval. The renewing nature of the flow also implies that  $\tilde{\mathcal{G}}$  depends only on the time difference  $\tau$  and not separately on the initial and final times in the interval  $[(n-1)\tau, n\tau]$ .

To obtain  $\tilde{\mathcal{G}}_{ij}(\mathbf{x}, \mathbf{x}_0, \mathbf{y}, \mathbf{y}_0, \tau)$  in the renewing flow, we use the method introduced by GB. The renewal time,  $\tau$ , is split into two equal sub-intervals. In the first sub-interval  $\tau/2$ , resistivity is neglected and the frozen field is advected with twice the original velocity. In the second sub-interval,  $\mathbf{u}$  is neglected and the field diffuses with twice the resistivity.

This method, plausible in the  $\tau \rightarrow 0$  limit, has been used to recover the standard mean field dynamo equations in renewing flows (Gilbert & Bayly 1992; Kolekar *et al.* 2012).

From the advective part of Eq. (2.1), we obtain the standard Cauchy solution, in the first sub-interval  $\tau/2 = t_1 - t_0$ ,

$$B_i(\mathbf{x}, t_1) = \frac{\partial x_i}{\partial x_{0j}} B_j(\mathbf{x}_0, t_0) \equiv J_{ij}(\mathbf{x}(\mathbf{x}_0)) B_j(\mathbf{x}_0, t_0). \quad (2.8)$$

Here  $B_j(\mathbf{x}_0, t_0)$  is the initial field, which propagates from  $\mathbf{x}_0$  at time  $t_0$ , to  $\mathbf{x}$  at time  $t_1 = t_0 + \tau/2$ . In Eq. (2.2), the phase  $\Phi = \mathbf{q} \cdot \mathbf{x} + \psi$  is constant in time as  $d\Phi/dt = \mathbf{q} \cdot \mathbf{u} = 0$ , from incompressibility. Then at time  $t_1 = t_0 + \tau/2$ , we integrate  $d\mathbf{x}/dt = 2\mathbf{u}$  to obtain,

$$\mathbf{x} = \mathbf{x}_0 + \tau\mathbf{u} = \mathbf{x}_0 + \tau\mathbf{a} \sin(\mathbf{q} \cdot \mathbf{x}_0 + \psi). \quad (2.9)$$

Thus the Jacobian is,

$$J_{ij}(\mathbf{x}(\mathbf{x}_0)) = \delta_{ij} + \tau a_i q_j \cos(\mathbf{q} \cdot \mathbf{x}_0 + \psi). \quad (2.10)$$

It will be more convenient to work with the resulting field in Fourier space,

$$\hat{B}_i(\mathbf{k}, t_1) = \int J_{ij}(\mathbf{x}(\mathbf{x}_0)) B_j(\mathbf{x}_0, t_0) e^{-i\mathbf{k} \cdot \mathbf{x}} d^3\mathbf{x}. \quad (2.11)$$

Then in the second sub-interval  $(t_1, t = t_1 + \tau/2)$ , only diffusion operates with resistivity  $2\eta$  to give,

$$\hat{B}_i(\mathbf{k}, t) = G^\eta(\mathbf{k}, \tau) \hat{B}_i(\mathbf{k}, t_1) = e^{-(\eta\tau\mathbf{k}^2)} \hat{B}_i(\mathbf{k}, t_1), \quad (2.12)$$

where  $G^\eta$  is the resistive Greens function. We combine Eq. (2.11) and Eq. (2.12) to derive the evolution equation for the magnetic two point correlation function,

$$\langle \hat{B}_i(\mathbf{k}, t) \hat{B}_h^*(\mathbf{p}, t) \rangle = e^{-\eta\tau(\mathbf{k}^2 + \mathbf{p}^2)} \int \left\langle J_{ij}(\mathbf{x}_0) J_{hl}(\mathbf{y}_0) e^{-i(\mathbf{k} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{y})} \right\rangle M_{jl}(\mathbf{r}_0, t_0) d^3\mathbf{x} d^3\mathbf{y}. \quad (2.13)$$

The statistical homogeneity of the field also implies the two-point magnetic correlator in Fourier space will be given by,

$$\langle \hat{B}_i(\mathbf{k}, t) \hat{B}_h^*(\mathbf{p}, t) \rangle = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{p}) \hat{M}_{ih}(\mathbf{p}, t). \quad (2.14)$$

We use Eq. (2.9) to transform from  $(\mathbf{x}, \mathbf{y})$  to  $(\mathbf{x}_0, \mathbf{y}_0)$  in Eq. (2.13). Due to incompressibility of the flow, the Jacobian of this transformation is unity. We also write  $\mathbf{k} \cdot \mathbf{x}_0 - \mathbf{p} \cdot \mathbf{y}_0 = \mathbf{k} \cdot \mathbf{r}_0 + \mathbf{y}_0 \cdot (\mathbf{k} - \mathbf{p})$  in Eq. (2.13), transform from  $(\mathbf{x}_0, \mathbf{y}_0)$  to a new set of variables  $(\mathbf{r}_0, \mathbf{y}_0' = \mathbf{y}_0)$ , and integrate over  $\mathbf{y}_0'$ . This leads to a delta function in  $(\mathbf{k} - \mathbf{p})$  and Eq. (2.13) becomes,

$$\begin{aligned} \hat{M}_{ih}(\mathbf{p}, t) &= e^{-2\eta\tau\mathbf{p}^2} \int \langle R_{ijhl} \rangle M_{jl}(\mathbf{r}_0, t_0) e^{-i\mathbf{p} \cdot \mathbf{r}_0} d^3\mathbf{r}_0 \\ \langle R_{ijhl} \rangle &= \left\langle J_{ij}(\mathbf{x}_0) J_{hl}(\mathbf{y}_0) e^{-i\tau(\mathbf{a} \cdot \mathbf{p})(\sin A - \sin B)} \right\rangle \end{aligned} \quad (2.15)$$

where,  $A = (\mathbf{x}_0 \cdot \mathbf{q} + \psi)$  and  $B = (\mathbf{y}_0 \cdot \mathbf{q} + \psi)$ . Due to statistical homogeneity of the renewing flow, we expect  $\langle R_{ijhl} \rangle$  to be only a function of  $\mathbf{r}_0$ , which we will see explicitly later.

### 3. The generalized Kazantsev equation

Exact evaluation of  $\langle R_{ijhl} \rangle$  is difficult. However, we can motivate a Taylor series expansion of the exponential in  $\langle R_{ijhl} \rangle$  for small Strouhl number  $St = q|\mathbf{a}|\tau = qa\tau$ , as follows.

Firstly in the argument of the exponential,  $(\sin A - \sin B) = \sin(\mathbf{q} \cdot \mathbf{r}_0/2) \cos(\psi + \mathbf{q} \cdot \mathbf{R}_0)$ , where  $\mathbf{R}_0 = (\mathbf{x}_0 + \mathbf{y}_0)/2$ . Also for the kinematic fluctuation dynamo, the magnetic correlation function peaks around the resistive scale  $r_0 = |\mathbf{r}_0| \sim 1/(qR_M^{1/2})$ , or the spectrum peaks around  $p \sim (qR_M^{1/2})$ . (Here  $p = |\mathbf{p}|$ .) Also  $R_M \sim a/(q\eta) \gg 1$ . Thus,  $qr_0 \ll 1$  and  $\sin(\mathbf{q} \cdot \mathbf{r}_0) \sim \mathbf{q} \cdot \mathbf{r}_0$ . Subsequently the phase of the exponential in Eq. (2.15) is of order  $(pa\tau qr_0) \sim qa\tau = St$ . Thus for  $St \ll 1$ , one can expand the exponential in Eq. (2.15) in  $\tau$ . We do this retaining terms up to  $\tau^4$  order; keeping up to  $\tau^2$  terms in Eq. (2.15), gives the Kazantsev equation, while the  $\tau^4$  terms give finite- $\tau$  corrections. We get,

$$\langle R_{ijhl} \rangle = \left\langle H_{ijhl} \left[ 1 - i\tau\sigma - \frac{\tau^2\sigma^2}{2!} + \frac{i\tau^3\sigma^3}{3!} + \frac{\tau^4\sigma^4}{4!} \right] \right\rangle, \quad (3.1)$$

where  $\sigma = (\mathbf{a} \cdot \mathbf{p})(\sin A - \sin B)$  and  $H_{ijhl} = J_{ij}(\mathbf{x}_0)J_{hl}(\mathbf{y}_0)$  contains terms up to order  $\tau^2$ . We note that Kleerorin *et al.* (2002) seem to have kept only up to  $p^2$  terms in Eq. (3.1).

### 3.1. Kazantsev equation from terms up to order $\tau^2$

We now consider all terms in Eq. (3.1) one by one up to the order  $\tau^2$  and average over  $\psi$ ,  $\hat{\mathbf{a}}$  and  $\hat{\mathbf{q}}$ . First consider  $\langle H_{ijhl} \rangle$  from Eq. (3.1),

$$\langle H_{ijhl} \rangle = \left\langle \delta_{ij}\delta_{hl} + \delta_{ij}a_hq_l \cos A + \delta_{hl}a_iq_j \cos B + a_ia_hq_jq_l \frac{\tau^2}{2} (\cos(\mathbf{q} \cdot \mathbf{r}_0) + \cos(2\mathbf{q} \cdot \mathbf{R}_0 + 2\psi)) \right\rangle \quad (3.2)$$

In Eq. (3.2), the second, third and last term on the right are proportional to  $\cos(\dots + n\psi)$  and hence go to zero on averaging over  $\psi$ . Survival of such terms which depend explicitly on  $\mathbf{x}_0$ ,  $\mathbf{y}_0$  or  $\mathbf{R}_0$  and would break statistical homogeneity. The resulting expression after averaging over  $\psi$  is,

$$\langle H_{ijhl} \rangle = \left\langle \delta_{ij}\delta_{hl} + a_ia_hq_jq_l \frac{\tau^2}{2} \cos(\mathbf{q} \cdot \mathbf{r}_0) \right\rangle = \delta_{ij}\delta_{hl} - \frac{\tau^2}{2} \partial_j \partial_l \langle a_ia_h \cos(\mathbf{q} \cdot \mathbf{r}_0) \rangle \quad (3.3)$$

where we have expressed  $q_j \cos(\mathbf{q} \cdot \mathbf{r}_0)$  as  $\partial_j \sin(\mathbf{q} \cdot \mathbf{r}_0)$ . We find that the expression in Eq. (3.3) contains the two point velocity correlator or the turbulent diffusion tensor, given by,

$$T_{ih} = \langle u_i(\mathbf{x}_0)u_h(\mathbf{y}_0) \rangle = \frac{\tau}{2} \langle a_ia_h \sin(A) \sin(B) \rangle = \frac{\tau}{4} \langle a_ia_h \cos(\mathbf{q} \cdot \mathbf{r}_0) \rangle. \quad (3.4)$$

Then we can express Eq. (3.3) as,

$$\langle H_{ijhl} \rangle = \delta_{ij}\delta_{hl} - 2\tau \partial_j \partial_l T_{ih}. \quad (3.5)$$

Consider now the second term in Eq. (3.1),  $i\tau \langle H_{ijhl} \sigma \rangle$ . We average over  $\psi$  and obtain statistically homogeneous terms,

$$\begin{aligned} \langle i\tau H_{ijhl} \sigma \rangle &= \frac{i\tau^2}{2} \langle \mathbf{a} \cdot \mathbf{p} [\delta_{ij} a_h q_l \sin(\mathbf{q} \cdot \mathbf{r}_0) + \delta_{hl} a_i q_j \sin(\mathbf{q} \cdot \mathbf{r}_0)] \rangle \\ &= \frac{-i\tau^2}{2} p_m [\delta_{ij} \partial_l \langle a_h a_m \cos(\mathbf{q} \cdot \mathbf{r}_0) \rangle + \delta_{hl} \partial_j \langle a_i a_m \cos(\mathbf{q} \cdot \mathbf{r}_0) \rangle] \\ &= -2i\tau p_m [\delta_{ij} \partial_l T_{hm} + \delta_{hl} \partial_j T_{im}] \end{aligned} \quad (3.6)$$

where again in the last equation we have identified and expressed in terms of the turbulent diffusion tensor. Similarly for the third term in Eq. (3.1) to order  $\tau^2$ ,

$$\begin{aligned} \left\langle H_{ijhl} \frac{\tau^2 \sigma^2}{2} \right\rangle &= \frac{\tau^2}{2} \delta_{ij} \delta_{hl} p_m p_n \langle a_m a_n [1 - \cos(\mathbf{q} \cdot \mathbf{r}_0)] \rangle \\ &= 2\tau \delta_{ij} \delta_{hl} p_m p_n [T_{mn}(0) - T_{mn}]. \end{aligned} \quad (3.7)$$

Now collecting all the simplified expressions of terms in Eq. (3.1) up to order  $\tau^2$ , as given in Eq. (3.5), Eq. (3.6) and Eq. (3.7), we obtain,

$$\begin{aligned} \langle R_{ijhl} \rangle &= \delta_{ij} \delta_{hl} - 2\tau \partial_j \partial_l T_{ih} - i2\tau p_m [\delta_{ij} \partial_l T_{hm} + \delta_{hl} \partial_j T_{im}] \\ &+ 2\tau \delta_{ij} \delta_{hl} p_m p_n [T_{mn}(0) - T_{mn}] \end{aligned} \quad (3.8)$$

We then substitute Eq. (3.8) into Eq. (2.15) and take the inverse Fourier transform of  $\hat{M}_{ih}(\mathbf{p}, t)$ .

$$M_{ih}(\mathbf{r}, t) = \int (1 - 2\eta\tau \mathbf{p}^2) \langle R_{ijhl} \rangle M_{jl}(\mathbf{r}_0, t_0) e^{-i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)} d^3 \mathbf{r}_0 \frac{d^3 \mathbf{p}}{(2\pi)^3} \quad (3.9)$$

where we also expand the exponential in the resistive Greens function and consider only leading order term in  $\eta$ , relevant in the independent small  $\eta$  (or  $R_M \gg 1$ ) limit. In Eq. (3.9), we consider only the first term in  $\langle R_{ijhl} \rangle$ ,  $\delta_{ij} \delta_{hl}$  to multiply with  $2\eta\tau \mathbf{p}^2$  since all the other terms will be of the order higher than  $\tau^2$ . In the case of the first two terms in  $\langle R_{ijhl} \rangle$  multiplying with unity, the integral in Eq. (3.9) is trivial with integration over  $\mathbf{p}$  first giving a delta function  $\delta^3(\mathbf{r} - \mathbf{r}_0)$  which then leads to all functions of  $\mathbf{r}_0$  simply turning into functions of  $\mathbf{r}$ , on integrating over  $\mathbf{r}_0$ . The other terms containing  $p_i$  can be first written as derivatives with respect to  $r_i$ . For example, consider the integral in Eq. (3.9) containing the third term in  $\langle R_{ijhl} \rangle$ ,

$$\begin{aligned} &\int 2\tau \delta_{ij} \delta_{hl} p_m p_n [T_{mn}(0) - T_{mn}] M_{jl}(\mathbf{r}_0, t_0) e^{-i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)} d^3 \mathbf{r}_0 \frac{d^3 \mathbf{p}}{(2\pi)^3} \\ &= \int 2\tau \left( \frac{-\partial_m}{i} \right) \left( \frac{-\partial_n}{i} \right) [T_{mn}(0) - T_{mn}] M_{ih}(\mathbf{r}_0, t_0) e^{-i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}_0)} d^3 \mathbf{r}_0 \frac{d^3 \mathbf{p}}{(2\pi)^3} \\ &= -2\tau \partial_m \partial_n [(T_L(0) - T_{mn}) M_{ih}(\mathbf{r}_0, t_0)] \end{aligned} \quad (3.10)$$

where we have used the fact that for a statistically homogeneous, isotropic and non helical velocity field, the correlation function

$$T_{ih} = (\delta_{ih} - \hat{r}_i \hat{r}_h) T_N(r, t) + \hat{r}_i \hat{r}_h T_L(r, t) \quad (3.11)$$

where  $\hat{r}_i = r_i/r$  and hence  $T_{mn}(0) = \delta_{mn} T_L(0)$ . Here  $T_L(r, t) = \hat{r}_i \hat{r}_h T_{ih}$  and  $T_N(r, t) = (1/2r)[\partial(r^2 T_L)/\partial r]$  are, respectively, the longitudinal and transversal correlation functions of the velocity field. Carrying out all the steps, and noting that  $(M_{ih}(\mathbf{r}, t) - M_{ih}(\mathbf{r}, t_0))/\tau = \partial M_{ih}/\partial t$  in the limit  $\tau \rightarrow 0$ , the resulting equation for  $M_{ih}$  is given by,

$$\begin{aligned} \frac{\partial M_{ih}(\mathbf{r}, t)}{\partial t} &= 2(-[T_{ih} M_{jl}]_{,jl} + [T_{mh} M_{il}]_{,ml} + [T_{im} M_{jh}]_{,jm} - [T_{mn} M_{ih}]_{,mn}) \\ &+ (2T_L(0) + 2\eta) \nabla^2 M_{ih} \end{aligned} \quad (3.12)$$

Note that we have statistically homogeneous, isotropic and non helical magnetic field, and hence similar to the velocity correlation function, we have  $M_{ih} = (\delta_{ih} - \hat{r}_i \hat{r}_h) M_N(r, t) + \hat{r}_i \hat{r}_h M_L(r, t)$ . Here  $M_L(r, t)$  and  $M_N(r, t)$  are, the longitudinal and transversal correlation functions of the magnetic field. Then on contracting Eq. (3.12) with  $\hat{r}_i \hat{r}_h$  we obtain the dynamical equation for  $M_L(r, t)$ , the Kazantsev equation. Note that we haven't yet performed averages over  $\mathbf{a}$  and  $\mathbf{q}$  because we have simply identified the two point velocity correlator from Eq. (3.4) in expressions evaluated after averaging over  $\psi$  (as in Eqs (3.5), (3.6) and (3.7)). We will explicitly have to perform the averages over  $\mathbf{a}$  and  $\mathbf{q}$  later when we obtain the dynamical equation for  $M_L$ .

3.2. Extending Kazantsev equation to higher order in  $\tau$ 

Next, we will consider the terms higher order in  $\tau$ , starting with  $\tau^3$  and then  $\tau^4$ . Interestingly, it turns out that all the terms of order  $\tau^3$  go to 0 on averaging. For example, from the second term in Eq. (3.1), we obtain,  $\tau^3 \langle i(\mathbf{p} \cdot \mathbf{a}) a_i a_h q_j q_l \cos A \cos B (\sin A - \sin B) \rangle$ . Here  $\cos A \cos B \sin A = (1/2) \sin(2A) \cos B = (1/4) [\sin(2A + B) - \sin(2A - B)]$ , contain  $\psi$  in their argument and hence go to 0 on averaging.

Now we consider the terms of order  $\tau^4$ . The first contribution is from the third term in Eq. (3.1),  $\tau^4 \langle -[(\mathbf{p} \cdot \mathbf{a})^2/2] a_i a_h q_j q_l \cos A \cos B [\sin A - \sin B]^2 \rangle$ . On averaging over  $\psi$ , we obtain,

$$\begin{aligned} & -\frac{\tau^4}{8} \langle a_i a_h (a_n p_n a_m p_m) q_j q_l [\cos(\mathbf{q} \cdot \mathbf{r}_0) - \cos(2\mathbf{q} \cdot \mathbf{r}_0)] \rangle \\ & = \frac{\tau^4}{8} \left\langle a_i a_h (a_n p_n a_m p_m) \partial_j \partial_l \left[ \cos(\mathbf{q} \cdot \mathbf{r}_0) - \frac{\cos(2\mathbf{q} \cdot \mathbf{r}_0)}{4} \right] \right\rangle \end{aligned} \quad (3.13)$$

We identify the terms in Eq. (3.13) with fourth order two point velocity correlators. Three of such velocity correlators can be defined,

$$\begin{aligned} T_{mni h}^{x^2 y^2} &= \tau^2 \langle u_m(\mathbf{x}) u_n(\mathbf{y}) u_i(\mathbf{x}) u_h(\mathbf{y}) \rangle, \\ T_{mni h}^{x^3 y} &= \tau^2 \langle u_m(\mathbf{x}) u_n(\mathbf{x}) u_i(\mathbf{x}) u_h(\mathbf{y}) \rangle, \\ T_{mni h}^{x^4} &= \tau^2 \langle u_m(\mathbf{x}) u_n(\mathbf{x}) u_i(\mathbf{x}) u_h(\mathbf{x}) \rangle. \end{aligned} \quad (3.14)$$

Again we multiply the fourth order velocity correlators by  $\tau^2$ , as we envisage that  $T_{ijkl}$  will be finite even in the  $\tau \rightarrow 0$  limit, behaving like products of turbulent diffusion. Note that the renewing flow is not Gaussian random, and hence higher order correlators of  $\mathbf{u}$  are not the product of two-point correlators. We consider the  $\psi$  averaging of the velocity correlators in Eq. (3.14), to obtain,

$$T_{mni h}^{x^2 y^2} = \tau^2 \langle a_m a_n a_i a_h \sin^2 A \sin^2 B \rangle = \frac{\tau^2}{4} \left\langle a_m a_n a_i a_h \left( 1 + \frac{\cos(2\mathbf{q} \cdot \mathbf{r}_0)}{2} \right) \right\rangle \quad (3.15)$$

$$T_{mni h}^{x^3 y} = \tau^2 \langle a_m a_n a_i a_h \sin^3 A \sin B \rangle = \frac{3\tau^2}{8} \langle a_m a_n a_i a_h \cos(\mathbf{q} \cdot \mathbf{r}_0) \rangle \quad (3.16)$$

$$T_{mni h}^{x^4} = \tau^2 \langle a_m a_n a_i a_h \sin^4 A \rangle = \frac{3\tau^2}{8} \langle a_m a_n a_i a_h \rangle \quad (3.17)$$

Now we can rewrite Eq. (3.13), by expressing it in terms of the velocity correlators we have obtained in Eqs (3.16) and (3.17). We have,

$$-\tau^2 p_n p_m \partial_j \partial_l \left[ \frac{T_{mni h}^{x^2 y^2}}{4} - \frac{T_{mni h}^{x^3 y}}{3} \right] \quad (3.18)$$

Note that the first term in Eq. (3.16) does not survive due to the derivatives in Eq. (3.18). Similarly from the fourth term in Eq. (3.1), the contribution of the order  $\tau^4$  is given by,

$$\begin{aligned} & i\tau^4 \frac{(\mathbf{p} \cdot \mathbf{a})^3}{6} [\delta_{ij} a_h q_l \cos B + \delta_{hl} a_i q_j \cos A] (\sin A - \sin B)^3 \\ & = i\frac{\tau^4}{8} p_n p_m p_r \left( \left\langle \delta_{ij} a_k a_n a_m a_r \partial_l \left[ 2 \sin(\mathbf{q} \cdot \mathbf{r}_0) - \frac{\sin(2\mathbf{q} \cdot \mathbf{r}_0)}{2} \right] \right\rangle \right) \\ & - \tau^2 2 p_n p_m p_r \left( \delta_{ij} \partial_l \left[ \frac{T_{mni h}^{x^2 y^2}}{4} - \frac{T_{mni h}^{x^3 y}}{3} \right] \right) \end{aligned} \quad (3.19)$$

where we have again expressed in terms of velocity correlators from Eqs (3.16) and (3.17).

Lastly, from the fifth term in Eq. (3.1),  $\frac{\tau^4}{24}\delta_{ij}\delta_{hl}\left\langle(\mathbf{p}\cdot\mathbf{a})^4(\sin A - \sin B)^4\right\rangle$  we have,

$$\begin{aligned} &= \frac{\tau^4}{16}\delta_{ij}\delta_{hl}p_m p_n p_r p_s \left\langle a_m a_n a_r a_s \left( \frac{3}{2} - 2\cos(\mathbf{q}\cdot\mathbf{r}_0) + \frac{\cos(2\mathbf{q}\cdot\mathbf{r}_0)}{2} \right) \right\rangle \\ &= \tau^2\delta_{ij}\delta_{hl}p_m p_n p_r p_s \left[ \frac{T_{mni h}^{x^2 y^2}}{4} - \frac{T_{mni h}^{x^3 y}}{3} + \frac{T_{mni h}^{x^4}}{12} \right] \end{aligned} \quad (3.20)$$

We again find that the integrand determining the magnetic spectral tensor  $\hat{M}_{ih}(\mathbf{p}, t)$ , is of the form  $G(\mathbf{p})F_{ih}(\mathbf{r}_0, t_0)$ , where  $G(\mathbf{p})$  is a polynomial up to second order in  $p_i$ . We can perform a simple inverse Fourier transform of  $\hat{M}_{ih}(\mathbf{p}, t)$ , in Eq. (2.15) back to configuration space and then magnetic field correlation function is,

$$M_{ih}(\mathbf{r}, t) = \int G(\mathbf{p})F_{ih}(\mathbf{r}_0, t_0)e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}_0)}d^3\mathbf{r}_0\frac{d^3\mathbf{p}}{(2\pi)^3}. \quad (3.21)$$

The  $p_i$  in  $G(\mathbf{p})$  above can be written as derivatives with respect to  $r_i$ . Then integral over  $\mathbf{p}$  simply gives a delta function  $\delta^3(\mathbf{r}-\mathbf{r}_0)$  and this makes the integral over  $\mathbf{r}_0$  trivial. This was explicitly demonstrated earlier in Eq. (3.10).

We then divide all the three contributions of the order  $\tau^4$  in Eqs (3.18), (3.19) and (3.20) by  $\tau$ . From the remaining  $\tau^3$ ,  $\tau^2$  is absorbed into  $T_{ijkl}$ , leaving one  $\tau$  which is treated as a small effective finite time parameter. Resulting extended equation for  $M_{ih}$  is given by,

$$\begin{aligned} \frac{\partial M_{ih}}{\partial t} &= 2(-[T_{ih}M_{jl}]_{,jl} + [T_{jh}M_{il}]_{,jl} + [T_{il}M_{jh}]_{,jl} - [T_{jl}M_{ih}]_{,jl}) + (2T_L(0) + 2\eta)\nabla^2 M_{ih} \\ &+ \tau \left( [\tilde{T}_{mni h}M_{jl}]_{,mnjl} - 2[\tilde{T}_{mnrh}M_{il}]_{,mnrl} + \left[ \left( \tilde{T}_{mnrs} + \frac{T_{mnrs}^{x^4}}{12} \right) M_{ih} \right]_{,mnrs} \right) \end{aligned} \quad (3.22)$$

where  $\tilde{T}_{mni h} = T_{mni h}^{x^2 y^2}/4 - T_{mni h}^{x^3 y}/3$ ,  $T_L(r) = \hat{r}_i \hat{r}_j T_{ij}$  with  $\hat{r}_i = r_i/r$ . The first line in Eq. (3.22) contains the terms which give the Kazantsev equation as in Eq. (3.12), while the second line contains the finite- $\tau$  corrections. We write these latter terms as fourth derivative of the combined velocity and magnetic correlators; however as both the velocity and magnetic fields are divergence free, each spatial derivative only acts on one or the other.

We then contract Eq. (3.22) with  $\hat{r}_i \hat{r}_h$  to obtain the dynamical equation for  $M_L(r, t)$ . On such a contraction, the terms in the first line lead to the original Kazantsev equation for  $M_L$ . In order to perform such a contraction, we need to know the explicit form of the fourth order velocity correlator,  $\tilde{T}_{mni h}$ . Such a fourth order two point correlator for a homogeneous and isotropic velocity field can be expressed as,

$$T_{mni h} = \hat{r}_{mni h} \bar{T}_L + \hat{P}_{(mn} \hat{P}_{ih)} \bar{T}_N + \hat{r}_{(mn} \hat{P}_{ih)} \bar{T}_{LN} \quad (3.23)$$

where  $\hat{r}_{mn} = \hat{r}_m \hat{r}_n$  and similarly  $\hat{r}_{mni h} = \hat{r}_m \hat{r}_n \hat{r}_i \hat{r}_h$ .  $\hat{P}_{mn} = \delta_{mn} - \hat{r}_{mn}$  is the configuration space projection operator.

$$\bar{T}_L = \hat{r}_{mni h} \tilde{T}_{mni h}, \quad \bar{T}_{LN} = \hat{r}_{mn} \hat{P}_{ih} \tilde{T}_{mni h}, \quad \bar{T}_N = \hat{P}_{mn} \hat{P}_{ih} \tilde{T}_{mni h}/16 \quad (3.24)$$

Lastly, the brackets  $()$  in the subscripts of two second rank tensors, denotes addition of all terms with all of the different permutations of the four indices considered in pairs. We will henceforth refer to all the ten terms in Eq. (3.23),  $\hat{r}_{mni h}, \hat{P}_{mn} \hat{P}_{ih}$  (and two other terms with permutations of the indices),  $\hat{r}_{(mn} \hat{P}_{ih)}$  (and five other terms with permutations of the



indices) as the basis tensors (Although not all of them are orthogonal to each other). For a divergence free (or incompressible) velocity field, the different correlation functions,  $\overline{T}_L$ ,  $\overline{T}_N$  and  $\overline{T}_{LN}$ , are related as,

$$\overline{T}_{LN} = \frac{1}{6r} \frac{d(r^2 \overline{T}_L)}{dr}, \quad \overline{T}_{LN} = \overline{T}_N + \frac{r}{4} \frac{d(\overline{T}_N)}{dr} \quad (3.25)$$

Consider the contraction of  $\hat{r}_{ih}$  with the first term in second line in Eq. (3.22),

$\hat{r}_{ih} [\tilde{T}_{mni h} M_{jl}]_{mnjl} = \hat{r}_{ih} \tilde{T}_{mni h, jl} M_{jl, mn}$ . Then we have,

$$\begin{aligned} \hat{r}_{ih} \tilde{T}_{mni h, jl} M_{jl, mn} = & \frac{1}{r^2} \left( [r_{ih} \tilde{T}_{mni h}]_{,jl} - [\delta_{ij} r_h \tilde{T}_{mni h}]_{,j} - [\delta_{il} r_h \tilde{T}_{mni h}]_{,l} \right. \\ & \left. - [\delta_{jh} r_i \tilde{T}_{mni h}]_{,j} - [\delta_{hl} r_i \tilde{T}_{mni h}]_{,j} + (\delta_{ij} \delta_{hl} + \delta_{il} \delta_{jh}) \tilde{T}_{mni h} \right) M_{jl, mn} \end{aligned} \quad (3.26)$$

We obtain a fourth order tensor from  $\hat{r}_{ih} \tilde{T}_{mni h, jl}$  which multiplies another fourth order tensor  $M_{jl, mn}$ . To make this computation tractable, we construct a table where we list the coefficients of all the basis tensors. We provide such a table in the Appendix A, (Table. 1). Similarly for the second term in second line in Eq. (3.22),

$$\begin{aligned} \hat{r}_{ih} [\tilde{T}_{mnrh} M_{il}]_{lmnr} = & (\hat{r}_h \tilde{T}_{mnrh, l}) (\hat{r}_i M_{il, mn}) = \frac{1}{r^2} \left( [r_h \tilde{T}_{mnrh}]_{,l} - \delta_{lh} \tilde{T}_{mnrh} \right) \times \\ & ([r_i M_{il}]_{,mn} - \delta_{ir} M_{il, mn} - \delta_{in} M_{il, mr} - \delta_{im} M_{il, nr}) \end{aligned} \quad (3.27)$$

Again we have given the expansion of the fourth order objects  $(\hat{r}_h \tilde{T}_{mnrh, l})$  and  $(\hat{r}_i M_{il, mn})$  (in terms of basis tensors), in Table. 2 in the Appendix A. Then lastly we have the third term from the second line in Eq. (3.22),

$$\begin{aligned} r_{ih} [\tilde{T}_{mnrs} M_{ih}]_{,mnrs} = & \tilde{T}_{mnrs} \left( [r_{ih} M_{ih}]_{,mnrs} - (r_{ih})_{,m} M_{ih, nrs} - (r_{ih})_{,n} M_{ih, mrs} \right. \\ & - (r_{ih})_{,r} M_{ih, nms} - (r_{ih})_{,s} M_{ih, nrm} - (r_{ih})_{mn} M_{ih, rs} - (r_{ih})_{mr} M_{ih, ns} \\ & \left. - (r_{ih})_{ms} M_{ih, rn} - (r_{ih})_{ns} M_{ih, mr} - (r_{ih})_{rs} M_{ih, mn} - (r_{ih})_{rn} M_{ih, ms} \right) \end{aligned} \quad (3.28)$$

Here the two fourth order tensor objects multiplying each other are  $\tilde{T}_{mnrs}$  and  $r_{ih} M_{ih, mnrs}$  and the expansion of such fourth order objects in terms of basis tensors can be again found in Table 3, in the Appendix A.

The tables 1, 2 and 3 are useful in making the algebra of all the fourth order terms in Eqs (3.26), (3.27) and (3.28) tractable. In each of the tables, we list the expansion of all the individual fourth order objects in terms of the basis tensors. The basis tensors form the rows, while the expansion coefficients in Eqs (3.26), (3.27) and (3.28) are listed as columns. Note that the first column is the list of the basis tensors. Then the subsequent columns list the expansion coefficients (of the respective basis tensor) for each fourth order terms in the Eqs (3.26), (3.27) and (3.28). Then we sum the contributions from each row, separately for the magnetic and velocity parts. The last but one column in Table 1 and the last columns in Table 2 and Table 3 give the resulting sum divided by  $r^2$ . We then finally multiply the sum obtained for the magnetic part with the sum from the velocity part.

Here, we note that when we multiply one group of the basis tensors with another, all of them go to zero, but yield a constant when multiplied within the same group. For example product of  $\hat{r}_{mni h}$  and  $\hat{r}_{mn} \hat{P}_{ih}$  goes to zero, but product of  $\hat{r}_{mni h}$  with itself naturally produces unity. Then the product of  $\hat{r}_{mn} \hat{P}_{ih}$  with  $\hat{r}_{ih} \hat{P}_{mn}$  (or the other four

similar kind of terms) goes to 0, but with itself gives a value of 2. Lastly, product of  $\hat{P}_{mn}\hat{P}_{ih}$  with  $\hat{P}_{mi}\hat{P}_{nh}$  (or  $\hat{P}_{ni}\hat{P}_{mh}$ ) gives a value of 2, but with itself gives a value of 4.

By multiplying the velocity part with the magnetic part in this manner, we finally obtain the additional terms from the contractions, due to finite  $\tau$  and extend the Kazantsev equation to the form,

$$\begin{aligned} \frac{\partial M_L(r, t)}{\partial t} &= \frac{2}{r^4} \frac{\partial}{\partial r} \left( r^4 \eta_{tot} \frac{\partial M_L}{\partial r} \right) + G M_L \\ &+ \tau M_L''' \left( \bar{T}_L + \frac{\bar{T}_L(0)}{12} \right) + \tau M_L''' \left( 2\bar{T}_L' + \frac{8\bar{T}_L}{r} + \frac{2\bar{T}_L(0)}{3r} \right) \\ &+ \tau M_L'' \left( \frac{5\bar{T}_L''}{3} + \frac{11\bar{T}_L'}{r} + \frac{8\bar{T}_L}{r^2} + \frac{2\bar{T}_L(0)}{3r^2} \right) \\ &+ \tau M_L' \left( \frac{2\bar{T}_L'''}{3} + \frac{17\bar{T}_L''}{3r} + \frac{5\bar{T}_L'}{r^2} - \frac{8\bar{T}_L}{r^3} - \frac{2\bar{T}_L(0)}{3r^3} \right) \end{aligned} \quad (3.29)$$

Here,  $\eta_{tot} = \eta + T_L(0) - T_L(r)$  and  $G = -2 \left( T_L'' + 4T_L'/r \right)$ . Here again the first line gives us the original Kazantsev equation and the rest of the terms form the extended part and have the parameter  $\tau$  multiplying them. We will refer to Eq. (3.29) as the generalized Kazantsev equation incorporating finite  $\tau$  effects. To proceed further, and solve the generalized Kazantsev equation Eq. (3.29) we need to firstly evaluate the second and fourth order velocity correlators explicitly for the renewing flow from Eq. (3.4) and Eqs (3.16), (3.17) and (3.17) respectively. Consider first the two point velocity correlator,

$$T_{ij} = \frac{\tau}{4} \langle A_l A_m P_{il} P_{jm} \cos(\mathbf{q} \cdot \mathbf{r}) \rangle = \frac{A^2 \tau}{12} \langle P_{ij} \cos(\mathbf{q} \cdot \mathbf{r}) \rangle = \frac{a^2 \tau}{8} \left[ \delta_{ij} + \frac{1}{q^2} \frac{\partial^2}{\partial r_i \partial r_j} \right] j_0(qr). \quad (3.30)$$

Here, we have made use of the results in Eqs (2.3) and (2.4), i.e. we have substituted for  $\mathbf{a}$  in terms of  $\mathbf{A}$ , and first averaged over  $\mathbf{A}$ . Similarly in the expression for  $T_{mnih}^{x^2 y^2}$  in Eq. (3.16), we substitute  $a_m = A_s \tilde{P}_{ms}$ ,  $a_n = A_t \tilde{P}_{nt}$ ,  $a_i = A_u \tilde{P}_{iu}$  and  $a_h = A_v \tilde{P}_{hv}$ . Then we have,

$$T_{mnih}^{x^2 y^2} = \frac{\tau^2 A^4}{60} \left\langle \tilde{P}_{(mn} \tilde{P}_{ih)} (1 + \cos(2\mathbf{q} \cdot \mathbf{r})) \right\rangle. \quad (3.31)$$

The first part in Eq. (3.31) is evaluated to be  $\langle \tilde{P}_{(mn} \tilde{P}_{ih)} \rangle = 8/15 (\delta_{(mn} \delta_{ih)})$ . And the second part in Eq. (3.31) is given as,

$$\begin{aligned} \left\langle \tilde{P}_{(mn} \tilde{P}_{ih)} \cos(2\mathbf{q} \cdot \mathbf{r}) \right\rangle &= [(\delta_{mn} + \partial_m \partial_n) (\delta_{ih} + \partial_i \partial_h) + (\delta_{mi} + \partial_m \partial_i) (\delta_{nh} + \partial_n \partial_h) \\ &\quad + (\delta_{mh} + \partial_m \partial_h) (\delta_{in} + \partial_i \partial_n)] j_0(2qr_0) \\ &= -24 \left( \frac{j_0(2z)}{(2z)^2} + \frac{3\partial_{2z} j_0(2z)}{(2z)^3} \right) \hat{r}_{mnih} + \left( j_0 + \frac{2\partial_{2z} j_0(2z)}{2z} - \frac{3\partial_{2z} j_0(2z)}{(2z)^2} - \frac{9\partial_{2z} j_0(2z)}{(2z)^3} \right) \\ &\quad \left[ \hat{P}_{(mn} \hat{P}_{ih)} \right] + \left( -\frac{4\partial_{2z} j_0(2z)}{z} + \frac{12\partial_{2z} j_0(2z)}{(2z)^2} + \frac{36\partial_{2z} j_0(2z)}{(2z)^3} \right) \left[ \hat{r}_{(mn} \hat{P}_{ih)} \right] \end{aligned} \quad (3.32)$$

where  $z = qr$  and the derivative  $\partial_{2z}$  is derivative with respect to  $2z$ . We get a similar expression as in Eq. (3.32) also for  $T_{mnih}^{x^3 y} = \frac{A^4}{40} \left\langle \tilde{P}_{(mn} \tilde{P}_{ih)} \cos(\mathbf{q} \cdot \mathbf{r}) \right\rangle$ , with all the  $(2z)$

replaced by  $z$  and  $\partial_{2z}$  by  $\partial_z$ . We give the expressions for  $\bar{T}_L^{x^2y^2}$  and  $\bar{T}_L^{x^3y}$ ,

$$\bar{T}_L^{x^2y^2} = \frac{-9a^4\tau^2}{10} \left( \frac{3\partial_{2z}j_0(2z)}{(2z)^3} + \frac{j_0(2z)}{(2z)^2} \right), \quad \bar{T}_L^{x^3y} = \frac{-27a^4\tau^2}{20} \left( \frac{3\partial_zj_0(z)}{z^3} + \frac{j_0(z)}{z^2} \right), \quad (3.33)$$

(The above expressions correct the missing  $\sim a^4\tau^2$  factors in Eq. (18) in BS14) These latter equalities give the explicit expressions of these fourth order correlators for the renewing flow. Eq. (3.29) allows eigen-solutions of the form  $M_L(z, t) = \tilde{M}_L(z)e^{\gamma\tilde{t}}$ , where  $\tilde{t} = t\eta_t q^2$ , with  $\eta_t = T_L(0) = a^2\tau/12 = A^2\tau/18$ , and  $\gamma$  is the growth rate. Boundary conditions are given as  $M_L'(0, t) = 0$ ,  $M_L \rightarrow 0$  as  $r \rightarrow \infty$ . Implications of the higher spatial derivative terms are discussed below.

#### 4. Growth rate and magnetic spectrum at finite correlation time

We now discuss the solution of Eq. (3.29) to examine the finite correlation time modification to the growth rate and magnetic correlation function or its energy spectrum. For the latter, we focus particularly on the large  $k$  (or small  $r$ ) behaviour. Recall that in the  $\tau \rightarrow 0$  limit the magnetic spectrum is of the Kazantsev form,  $M(k) \propto k^{3/2}$  for  $q \ll k \ll k_\eta$ . Our aim is to determine how this gets modified in the presence of finite correlation time effects. For this purpose, we employ two different approaches. First, we recall in more detail the scaling solution discussed in BS14. We also then present a WKBJ analysis to derive  $M_L(r, t)$  in the small  $r$  limit, and hence the magnetic spectrum.

In both approaches, to derive the standard Kazantsev spectrum in the large  $k$  limit, and its finite- $\tau$  modifications, it suffices to go to the limit of small  $z = qr \ll 1$ . Expanding the Bessel functions in Eqs (3.30) and (3.33) in this limit, and substituting  $M_L(z, t) = \tilde{M}_L(z)e^{\gamma\tilde{t}}$ , Eq. (3.29) becomes,

$$\begin{aligned} \gamma\tilde{M}_L(z) = & \left( \frac{2\eta}{\eta_t} + \frac{z^2}{5} \right) \tilde{M}_L'' + \left( \frac{8\eta}{\eta_t} + \frac{6z^2}{5} \right) \frac{\tilde{M}_L'}{z} + 2\tilde{M}_L \\ & + \frac{9\bar{\tau}}{175} \left( \frac{z^4}{2} \tilde{M}_L'''' + 8z^3 \tilde{M}_L''' + 36z^2 \tilde{M}_L'' + 48z \tilde{M}_L' \right) \end{aligned} \quad (4.1)$$

where  $\bar{\tau} = \tau\eta_t q^2 = (St)^2/12$  and prime is now  $z$ -derivative.

For the solution near the origin, where  $z \ll \sqrt{\eta/\eta_t}$ , it suffices to approximate  $\tilde{M}_L$  as a parabola and write  $\tilde{M}_L(z) = M_0(1 - z^2/z_\eta^2)$ . From Eq. (4.1), we find  $z_\eta = qr_\eta = [240/(2 - \gamma)]^{1/2} [R_M(St)]^{-1/2}$ . The  $\bar{\tau}$  dependent terms, which are small because both  $z$  and  $\bar{\tau}$  are small, do not affect this result. Thus for  $R_M \gg 1$ , the resistive scale  $r_\eta \ll 1/q$  (or  $k_\eta = 1/r_\eta \gg q$ ), although one has to go to sufficiently large  $R_M \gg 240/((2 - \gamma)St)$  for this conclusion to obtain.

In order to determine the magnetic correlation function for spatial scales larger than  $z_\eta$ , and also obtain the growth rate, we have to more fully analyze Eq. (4.1). We see that this evolution equation (or Eq. (3.29)), also has higher order (third and fourth) spatial derivatives when going to finite- $\tau$  case. This indicates that for finite  $\tau$ ,  $M_L$  evolution is actually nonlocal, determined by an integral type equation; but whose leading approximation for small  $\bar{\tau}$  is the local equation Eq. (4.1). However these higher derivative terms only appear as perturbative terms multiplied by the small parameter  $\bar{\tau}$ . Then it is possible to use the Landau-Lifshitz type approximation, earlier used in treating the effect of radiation reaction force in electrodynamics (see Landau & Lifshitz (1975) section 75). In this treatment, one first ignores the perturbative terms proportional to  $\bar{\tau}$ , which gives

basically Kazantsev equation for  $\tilde{M}_L$ , and uses this to express  $\tilde{M}_L'''$  and  $\tilde{M}_L''''$  in terms of the lower order derivatives  $\tilde{M}_L''$  and  $\tilde{M}_L'$ .

We will find that for both the scaling solution and for determining the asymptotic WKB solution, these higher order derivatives are only required in the limit  $z \gg z_\eta$ . In this limit we have from Eq. (4.1) at the zeroth order in  $\bar{\tau}$ ,

$$\frac{z^2}{5}\tilde{M}_L'' = -\frac{6z}{5}\tilde{M}_L' + (\gamma - 2)\tilde{M}_L \quad (4.2)$$

Differentiating this expression first once and then twice gives,

$$z^3\tilde{M}_L''' = -8z^2\tilde{M}_L'' - z(16 - 5\gamma_0)\tilde{M}_L', \quad z^4\tilde{M}_L'''' = (56 + 5\gamma_0)z^2\tilde{M}_L'' + 10(16 - 5\gamma_0)z\tilde{M}_L'. \quad (4.3)$$

Here  $\gamma_0$  is the growth rate which obtains for the Kazantsev equation in the  $\tau \rightarrow 0$  limit. We now turn to the scaling solution approach.

#### 4.1. Growth rate and magnetic correlations from a scaling solution

Consider the solution for  $z_\eta \ll z \ll 1$ . In this limit, ignoring terms depending on  $\eta/\eta_t$ , Eq. (4.1) itself is scale free, as scaling  $z \rightarrow cz$  leaves it invariant. Thus the resulting equation has power law solutions of the form  $\tilde{M}(z) = \bar{M}_0 z^{-\lambda}$ . To find the form of this solution, we first substitute the expressions in Eq. (4.3) back into the full Eq. (4.1). We get after neglecting the  $\eta/\eta_t$  terms,

$$\tilde{M}_L'' z^2 \left( \bar{\tau} \gamma_0 \frac{9}{70} + \frac{1}{5} \right) + \tilde{M}_L' z \left( \bar{\tau} \gamma_0 \frac{27}{35} + \frac{6}{5} \right) + (2 - \gamma)\tilde{M}_L = 0 \quad (4.4)$$

We find the interesting result that the coefficients of the perturbative terms in Eq. (4.1) are such that all perturbative terms which do not depend on  $\gamma_0$  cancel out in Eq. (4.4) !

As advertised Eq. (4.4) admits power law solutions of the form  $\tilde{M}_L(z) = \bar{M}_0 z^{-\lambda}$ , with  $\lambda$  determined by,

$$\lambda^2 - 5\lambda + \frac{5(2 - \gamma)}{1 + \frac{9}{14}\gamma_0\bar{\tau}} = 0; \quad \text{so } \lambda = \frac{5}{2} \pm i\lambda_I, \quad \lambda_I = \frac{1}{2} \left[ \frac{20(2 - \gamma)}{(1 + 9\gamma_0\bar{\tau}/14)} - 25 \right]^{1/2} \quad (4.5)$$

More important is the fact that the real part of  $\lambda$  is  $\lambda_R = 5/2$ , independent of the value of  $\bar{\tau}$ ! We can also get the approximate growth rate assuming  $R_M \gg 1$ , following an argument from Gruzinov *et al.* (1996). These authors looked at Eq. (4.5) as an equation for  $\gamma(\lambda)$  and argued that the growth rate is determined by substituting in to Eq. (4.5), the value of  $\lambda = \lambda_m$  where  $d\gamma/d\lambda = 0$ . This gives

$$\gamma_0 \approx 3/4, \quad \text{and} \quad \gamma \approx (3/4)(1 - (45/56)\bar{\tau}). \quad (4.6)$$

Note that Eq. (4.6) also implies  $\lambda_I \approx 0$ . (Including the effects of resistivity gives  $\lambda_I$ , a small positive non zero value  $\propto 1/(\ln(R_M))$  as will be shown below). The  $\gamma_0$  we get matches with that of Kulsrud & Anderson (1992), obtained from the evolution equation of  $M(k, t)$ . It is also important to note that the growth rate is reduced for a finite  $\bar{\tau}$ . This was found in simulations which directly compare with an equivalent Kazantsev model (Mason *et al.* 2011).

The form of the magnetic correlation  $M_L$  for  $z_\eta \ll z \ll 1$  can also be found from Eq. (4.5). It is given by

$$M_L(z, t) = e^{\gamma \bar{t}} \bar{M}_0 z^{-5/2} \sin(\lambda_I \ln(z) + \phi), \quad (4.7)$$

where  $\bar{M}_0$  and  $\phi$  are constants. Thus in this range,  $M_L$  varies dominantly as  $z^{-5/2}$ , modulated by the weakly varying sine factor (as  $\lambda_I$  is small). We will use this below

to determine the asymptotic magnetic spectrum. Before that, we turn to the alternate approach to determining  $\gamma$  and  $M_L$ , using the WKBJ approximation, which also allows one to incorporate the effects of the small resistive terms.

#### 4.2. Growth rate and Magnetic correlation function using WKBJ analysis

First it is convenient to define a scaled co-ordinate  $\bar{z} = (\sqrt{\eta_t/\eta}) z$ . In terms of this new coordinate the resistive scale will have  $\bar{z} \sim 1$ , where as the forcing scale,  $z = 1$  corresponds to  $\bar{z} \sim \sqrt{R_M} \gg 1$ . Now substituting the expressions in Eq. (4.3) back into the full Eq. (4.1) we get,

$$\frac{d^2 \tilde{M}_L}{d\bar{z}^2} \left( 2 + \bar{\tau}\gamma_0 \frac{9\bar{z}^2}{70} + \frac{\bar{z}^2}{5} \right) + \frac{d\tilde{M}_L}{d\bar{z}} \left( \frac{8}{\bar{z}} + \bar{\tau}\gamma_0 \frac{27\bar{z}}{35} + \frac{6\bar{z}}{5} \right) + (2 - \gamma)\tilde{M}_L = 0 \quad (4.8)$$

As remarked earlier, the coefficients of the perturbative terms in Eq. (4.1) are such that all perturbative terms which do not depend on  $\gamma_0$  cancel out in Eq. (4.8).

Further, in order to implement the boundary condition at  $\bar{z} = 0$ , under WKBJ approximation, it is better to transform to a new variable  $x$ , where  $\bar{z} = e^x$ . Also to eliminate first derivative terms in the resulting equation we substitute  $\tilde{M}_L(x) = g(x)W(x)$ , and choose  $g(x)$  to satisfy the differential equation,

$$\frac{1}{g} \frac{dg}{dx} = -\frac{5}{2} \frac{(6 + \bar{z}^2 F)}{(10 + \bar{z}^2 F)}, \quad \text{with } F = (1 + (9/14)\bar{\tau}\gamma_0). \quad (4.9)$$

Then  $W$  satisfies,

$$\frac{d^2 W}{dx^2} + p(x)W = 0 \quad (4.10)$$

where

$$p(x) = \frac{A_0 \bar{z}^4 - B_0 \bar{z}^2 - 225}{(10 + F\bar{z}^2)^2}, \quad (4.11)$$

$$A_0 = 5F \left( \frac{3}{4} - \frac{45}{56}\bar{\tau}\gamma_0 - \gamma \right), \quad B_0 = 5 \left( 10\gamma + \frac{171}{14}\bar{\tau}\gamma_0 - 1 \right). \quad (4.12)$$

The WKBJ solutions to this equation are linear combinations of

$$W = \frac{1}{p^{1/4}} \exp(\pm i \int^x p^{1/2} dx) \quad (4.13)$$

Note that as  $\bar{z} \rightarrow 0$ ,  $x \rightarrow -\infty$  and  $p \rightarrow -9/4$ ; so the WKBJ solutions are in the form of growing and decaying exponentials at this end. And as  $\bar{z}$  increases to a large enough value,  $p(x)$  goes through a zero at say  $\bar{z} = \bar{z}_0$  (or  $x = x_0$ ) and becomes positive for  $\bar{z} > \bar{z}_0$ . The solution then becomes oscillatory. Note that at  $\bar{z} \rightarrow +\infty$ , one would again want to solution to decay, and so  $p(x)$  should become negative. This cannot be seen in Eq. (4.11), as it is valid only for  $z \ll 1$  (or  $\bar{z} \ll \sqrt{R_M}$ ), but would require one to consider Eq. (3.29) in the opposite limit of  $z \gg 1$  (or  $\bar{z} \gg \sqrt{R_M}$ ). In such a limit one has  $T_L(r) \rightarrow 0$ ,  $\bar{T}_L(r) \rightarrow 0$ , and again using the Landau-Lifshitz ansatz to eliminate  $\tilde{M}_L''''$ ,  $\tilde{M}_L(z)$  now satisfies

$$\gamma \tilde{M}_L(z) = \left( \frac{2\eta}{\eta_t} + 2 + \bar{\tau}\alpha \right) \tilde{M}_L'' + 8 \left( \frac{\eta}{\eta_t} + 1 \right) \frac{\tilde{M}_L'}{z}, \quad (4.14)$$

where  $\alpha = (q^2 \bar{T}_L(0)\gamma_0)/[12(\eta + \eta_t)]$ . We can again transform to the  $x$ -coordinate, and write  $\tilde{M}_L = gW$ . Then in this limit of  $z \gg 1$ ,  $W$  again satisfies Eq. (4.10) with now

$$\frac{1}{g} \frac{dg}{dx} = -e^x \frac{(1 + \eta_t/\eta)}{2(2 + 2\eta_t/\eta + \bar{\tau}\alpha)}, \quad p(x) = -e^{2x} \frac{(1 + \eta_t/\eta) + \gamma}{(2 + 2\eta_t/\eta + \bar{\tau}\alpha)^2}. \quad (4.15)$$

We see that  $p(x)$  is now negative definite and so again one has exponentially damped solutions for  $W$ . Since  $p(x) > 0$  for  $\bar{z} > \bar{z}_0$ , and is negative at  $\bar{z} \gg \sqrt{R_M}$ , there would again be a point, say  $\bar{z} = \bar{z}_c$  (or  $x = x_c$ ), where it would go to zero. We approximate our WKB treatment by assuming that Eq. (4.8) is valid for  $z < 1$  and Eq. (4.14) is valid for  $z > 1$ . The outer transition point  $\bar{z}_c$  then can be taken to be the boundary between these two regions. We will see that the  $\bar{z}_c$  dependence, in the determination of the growth rate and  $M_L$  only comes within a logarithm, and so our results are not very sensitive to its exact value. This insensitivity to the outer boundary condition has been remarked earlier by several authors (Kulsrud & Anderson 1992; Gruzinov *et al.* 1996; Schekochihin *et al.* 2002; Brandenburg & Subramanian 2005).

The requirement that the oscillatory solution in the region  $\bar{z}_0 < \bar{z} < \bar{z}_c$  match on to the growing exponential near  $\bar{z} \ll \bar{z}_0$  and the decaying exponential as  $\bar{z} \gg \bar{z}_c$ , gives the standard condition (Bender & Orszag 1978; Mestel & Subramanian 1991; Subramanian 1997) on the the eigenvalue  $\gamma$

$$\int_{x_0}^{x_c} p^{1/2}(x) dx = \frac{(2n+1)\pi}{2}. \quad (4.16)$$

We will find that  $\bar{z}_0$  is large enough that one can neglect the constant terms in Eq. (4.11). Then the integral in Eq. (4.16) can be done exactly and leads to the condition,

$$A_0^{1/2} \left[ \ln \left( \frac{\bar{z}_c}{\bar{z}_0} + \left( \frac{\bar{z}_c^2}{\bar{z}_0^2} - 1 \right)^{1/2} \right) - \left( 1 - \frac{\bar{z}_0^2}{\bar{z}_c^2} \right)^{1/2} \right] = \frac{\pi F}{2}. \quad (4.17)$$

Here we have taken  $n = 0$  which corresponds to the fastest growing eigenfunction. We will also find self-consistently that for large  $R_M$ ,  $\bar{z}_c^2/\bar{z}_0^2 \gg 1$ . In this case Eq. (4.17) gives for the growth rate,

$$\gamma = \frac{3}{4} - \frac{45}{56} \bar{\tau} \gamma_0 - \frac{\pi^2}{5} \frac{(1 + (9/14) \bar{\tau} \gamma_0)}{(\ln(2\bar{z}_c/\bar{z}_0))^2} \approx \frac{3}{4} \left[ 1 - \frac{45}{56} \bar{\tau} \right] - \frac{\pi^2}{5} \frac{(1 + (27/56) \bar{\tau})}{(\ln(R_M))^2}. \quad (4.18)$$

In the latter part of Eq. (4.18), we have used self-consistent estimates of  $\gamma_0 \sim 3/4$ ,  $\bar{z}_c \sim \sqrt{\eta_t/\eta z_c} \sim \sqrt{R_M}$  and  $\bar{z}_0 \sim \sqrt{B_0/A_0} \sim \ln(R_M)$ , and so also neglected  $\ln \bar{z}_0$  compared to  $\ln \bar{z}_c$ . This result for the growth rate exactly matches with that obtained earlier by BS14 in the limit of large  $R_M$  using a scaling solution (see Eq. (4.6) above). It of course corrects this estimate for finite  $R_M$ . We also see from Eq. (4.18) that the growth rate is insensitive (more correctly only logarithmically sensitive) to the exact value of  $\bar{z}_c$ , the upper zero of  $p(x)$ .

The WKB analysis also gives the form of the eigenfunction between the two zeros

$$W(x) \approx \frac{1}{p^{1/4}} \sin \left[ \int_{x_1}^x (p)^{1/2} dx + \frac{\pi}{4} \right] \approx \frac{(\ln R_M)^{1/2}}{\pi^{1/2}} \sin \left[ \frac{\pi}{\ln R_M} \ln \left( \frac{\bar{z}}{\bar{z}_0} \right) + \frac{\pi}{4} \right] \quad (4.19)$$

where for the latter expression we have taken the large  $\bar{z} > \bar{z}_0 \gg 1$  limit which is applicable here. Also for  $\bar{z} \gg 1$ , we can see from Eq. (4.9) that  $(1/g)(dg/dx) \rightarrow -5/2$  independent of the value of  $\bar{\tau}$ . Thus in this limit  $g(x) \propto \exp(-5x/2)$ . Since  $M_L(z) \propto e^{\gamma \bar{t}} gW$ , the WKB solution for the region  $z_\eta \ll z \ll 1$  is then given by,

$$M_L(z, t) = e^{\gamma \bar{t}} \tilde{M}_0 z^{-5/2} \sin \left[ \frac{\pi}{\ln R_M} \ln \left( \frac{z}{z_0} \right) + \frac{\pi}{4} \right] \quad (4.20)$$

This again matches with the result obtained from the scaling solution, improving it by fixing the constants there, in particular  $\lambda_I$ . We see that the dominant variation of  $M_L(z, t)$

in this regime is the power law behaviour  $M_L \propto z^{-5/2}$ , modulated by the weakly varying sine factor, as before.

The power law scaling of the magnetic correlation function can be translated to the scaling of the magnetic power spectrum. It is straightforward to show that the magnetic power spectrum is related to the longitudinal correlation function  $M_L$  by (cf. Brandenburg & Subramanian (2000)),

$$M(k, t) = \int dr (kr)^3 M_L(r, t) j_1(kr) \quad (4.21)$$

The spherical Bessel function  $j_1(kr)$  is peaked around  $k \sim 1/r$ , and every value of  $k$  in  $M(k, t)$  gets dominant contribution in the integral in Eq. (4.21) from values of  $r \sim 1/k$ . Therefore a power law behaviour of  $M_L \propto z^{-\lambda_R}$  for a range of  $z_\eta \ll z = qr \ll 1$ , translates into a power law for the spectrum  $M(k) \propto k^{\lambda_R-1}$  in the corresponding wavenumber range  $q \ll k \ll q/z_\eta$ . Both the scaling solution in Eq. (4.7) and the WKBJ solution given in Eq. (4.20), show that in the range  $z_\eta \ll z \ll 1$ ,  $M_L$  dominantly varies as a power law with  $\lambda_R = 5/2$ , independent of  $\tau$ . This then leads to the remarkable result emphasized by BS14 that the magnetic spectrum is of the Kazantsev form with  $M(k) \propto k^{3/2}$  in  $k$ -space, independent of  $\tau$ !

## 5. Discussion and conclusions

Fluctuation dynamos, generic to any turbulent plasma, are likely to be crucial for rapid generation of magnetic fields in astrophysical systems. We have given here an analytical treatment of fluctuation dynamos at finite correlation times, by modelling the velocity as a flow which renews itself after every time step  $\tau$ . In particular we present a detailed derivation of the evolution equation for the two-point magnetic correlation function in such a flow, earlier spelled out briefly in BS14. This generalizes the Kazantsev equation which was derived under the assumption that the velocity is delta-correlated in time, to the situation where the correlation time is finite. The correlation time will indeed be finite in any turbulent flow. Our generalized evolution equation for  $M_L(r, t)$  (Eq. (3.29)), reduces to the Kazantsev equation when  $\tau \rightarrow 0$ , and extends it to the next order in  $\tau$ .

The evolution equation for such a finite  $\tau$ , involves both higher (fourth) order velocity correlators and also higher order (third and fourth) spatial derivatives of  $M_L$ , signalling that non-local effects are important in this case. However these higher order derivatives appear only perturbatively, multiplied by the small parameter  $\bar{\tau} = \tau \eta_t q^2$ . This allows us to use the Landau-Lifshitz approach, earlier used to treat the effect of the radiation reaction force in electrodynamics. In this approach, to the zeroth order in  $\bar{\tau}$ , one retains the standard Kazantsev equation. This is then used to express the third and fourth derivatives of  $M_L$  in terms of the lower order derivatives, to finally get an evolution equation which at most involves second derivatives of  $M_L$ .

The resulting evolution equation is analyzed both using a scaling solution and the WKBJ approximation. The scaling solution is valid in the range of scales, where resistivity can be neglected, while the WKBJ treatment also takes into account the effect of a finite resistivity. From both treatments we see that the effect of a finite  $\tau$  is to cause a reduction in the dynamo growth rate. The asymptotic form of the correlation function on scales  $z_\eta \ll z \ll 1/q$  is very nearly a power law,  $M_L \propto z^{-5/2}$  independent of  $\tau$ ! This leads to the important and intriguing result that the Kazantsev spectrum of  $M(k) \propto k^{3/2}$ , is preserved even at finite- $\tau$ .

Although we derived the effects of a finite- $\tau$  using a particular renewing velocity field, the resulting evolution equation for  $M_{ih}$  (Eq. (3.22)) or  $M_L$  (Eq. (3.29)), can be cast

completely in terms of the general velocity correlators,  $T_{ij}$  and  $T_{ijkl}$ . It also matches exactly with Kazantsev equation for the  $\tau \rightarrow 0$  case. Moreover, we expect the forms of  $T_{ij}$  and  $T_{ijkl}$  at  $r \ll 1/q$ , to be universal due to their symmetries and divergence free properties. We would therefore conjecture that our results on the magnetic spectrum could have a more general validity than the context (of a renewing velocity) in which it is derived. Future work would involve a numerical study of Eq. (3.29) without making the small  $z$  approximation. The general methodology developed here also hold the promise of being systematically extendable to the non-perturbative regime of  $St \sim 1$ , at least by a series of numerical integrations to implement the averaging. The inclusion of shear and helicity are also the next obvious extensions that need to be studied, issues which we hope to address in the future.

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## Appendix A. Tables for tracking isotropic and homogeneous fourth order tensors

TABLE 1. The basis tensor components for all fourth order tensors involved in Eq. (3.26)

Terms	$\left[ r_{ih} \tilde{T}_{mni h} \right]_{,jl}$	$\left[ \begin{array}{l} -\left[ \delta_{ij} r_h \tilde{T}_{mni h} \right]_{,l} \\ -\left[ \delta_{hj} r_i \tilde{T}_{mni h} \right]_{,l} \end{array} \right] =$	$\left[ \begin{array}{l} -\left[ \delta_{il} r_h \tilde{T}_{mni h} \right]_{,j} \\ -\left[ \delta_{hl} r_i \tilde{T}_{mni h} \right]_{,j} \end{array} \right] =$	$(\delta_{ij} \delta_{hl}) \tilde{T}_{mni h}$	Sum/ $r^2$	$M_{jl, mn}$
$r_{jlmn}$	$\frac{r^2 \bar{T}_L''}{4 \bar{T}_L' r + 2 \bar{T}_L}$	$-\bar{T}_L' r - \bar{T}_L$	$-\bar{T}_L' r - \bar{T}_L$	$2 \bar{T}_L$	$\bar{T}_L''$	$M_L''$
$\hat{P}_{jlmn}$	$\bar{T}_L' r + 2 \bar{T}_N$	$-\bar{T}_L + 2 \bar{T}_N$	$-\bar{T}_L + 2 \bar{T}_N$	$2 \bar{T}_N$	$\frac{\bar{T}_L'}{r} - \frac{(4 \bar{T}_L - 12 \bar{T}_N)}{r^2}$	$2 M_L'' + \frac{r M_L'''}{2}$
$\hat{P}_{ml} r_{jn},$ $\hat{P}_{nl} r_{mj}$	$\frac{(\bar{T}_L' - \bar{T}_N') r + (\bar{T}_L - \bar{T}_N)}{(\bar{T}_L - \bar{T}_N)}$	$-\bar{T}_L + 2 \bar{T}_N$	$-\bar{T}_N' r - \bar{T}_N$	$2 \bar{T}_N$	$\frac{(\bar{T}_L' - 3 \bar{T}_N')}{r} - \frac{(\bar{T}_L - 3 \bar{T}_N)}{r^2}$	$\frac{-M_L''}{2}$
$\hat{P}_{nj} r_{ml},$ $\hat{P}_{mj} r_{ln}$	$\frac{(\bar{T}_L' - \bar{T}_N') r + (\bar{T}_L - \bar{T}_N)}{(\bar{T}_L - \bar{T}_N)}$	$-\bar{T}_N' r - \bar{T}_N$	$-\bar{T}_L + 2 \bar{T}_N$	$2 \bar{T}_N$	$\frac{(\bar{T}_L' - 3 \bar{T}_N')}{r} - \frac{(\bar{T}_L - 3 \bar{T}_N)}{r^2}$	$\frac{-M_L''}{2}$
$\hat{P}_{mn} r_{jl}$	$\frac{\bar{T}_N'' r^2 + 4 \bar{T}_N' r + 2 \bar{T}_N}{4 \bar{T}_N' r + 2 \bar{T}_N}$	$-\bar{T}_N' r - \bar{T}_N$	$-\bar{T}_N' r - \bar{T}_N$	$2 \bar{T}_N$	$\bar{T}_N''$	$\frac{2 M_L'}{r}$
$\hat{P}_{jl} \hat{P}_{mn}$	$\bar{T}_N' r + 2 \bar{T}_N$	$-\bar{T}_N$	$-\bar{T}_N$	$2 \bar{T}_{LN}$	$\frac{\bar{T}_N'}{r} + \frac{2(\bar{T}_{LN} - \bar{T}_N)}{r^2}$	$\frac{3 M_L'}{2 r} + \frac{M_L''}{2}$
$\hat{P}_{mj} \hat{P}_{ln},$ $\hat{P}_{nj} \hat{P}_{lm}$	$(\bar{T}_L - \bar{T}_N)$	$-\bar{T}_N$	$-\bar{T}_N$	$2 \bar{T}_{LN}$	$\frac{\bar{T}_L' - 5 \bar{T}_N}{r^2} + \frac{(2 \bar{T}_{LN})}{r^2}$	$\frac{-M_L'}{2 r}$

TABLE 2. The basis tensor components for all fourth order tensors involved in Eq. (3.27)

Terms	$[r_h T_{hnmr}]_{,l}$	$-\delta_{lh} T_{hnmr}$	Sum/ $r^2$
$r_{lmnr}$	$ \overline{T}_L' r + \overline{T}_L $	$ \overline{T}_L $	$ \frac{\overline{T}_L'}{r} $
$\hat{P}_{ln} r_{mr}, \hat{P}_{lr} r_{nm}, \hat{P}_{lm} r_{rn}$	$ \overline{T}_L - 2\overline{T}_N $	$ \overline{T}_N $	$ \frac{(\overline{T}_L - 3\overline{T}_N)}{r^2} $
$\hat{P}_{mr} r_{ln}, \hat{P}_{mn} r_{lr}, \hat{P}_{rn} r_{lm}$	$ \overline{T}_N' r + \overline{T}_N $	$ \overline{T}_N $	$ \frac{\overline{T}_N'}{r^2} $
$\hat{P}_{mr} \hat{P}_{ln}, \hat{P}_{mn} \hat{P}_{lr}, \hat{P}_{rn} \hat{P}_{lm}$	$ \overline{T}_N $	$ \overline{T}_{LN} $	$ \frac{(-\overline{T}_{LN} + \overline{T}_N)}{r^2} $

Terms	$[r_j M_{jl}]_{,rmn}$	$-(\delta_{jr} M_{jl,mn})$	$-(\delta_{jn} M_{jl,mr})$	$-(\delta_{jm} M_{jl,rn})$	Sum/ $r^2$
$r_{lmnr}$	$ M_L''' r + 3M_L $	$ M_L'' $	$ M_L'' $	$ M_L'' $	$ r M_L''' $
$\hat{P}_{ln} r_{mr}$	$ M_L'' $	$ \frac{M_L''}{2} $	$ -2M_L'' - \frac{M_L''}{2} r $	$ \frac{M_L''}{2} $	$ \frac{M_L''}{2} $
$\hat{P}_{mr} r_{ln}$	$ M_L'' $	$ \frac{M_L''}{2} $	$ \frac{-2M_L'}{r} $	$ \frac{M_L''}{2} $	$ 2M_L'' - \frac{2M_L'}{r} $
$\hat{P}_{lm} r_{nr}$	$ M_L'' $	$ \frac{M_L''}{2} $	$ \frac{M_L''}{2} $	$ -2M_L'' - \frac{M_L''}{2} r $	$ \frac{M_L''}{2} $
$\hat{P}_{nr} r_{lm}$	$ M_L'' $	$ \frac{M_L''}{2} $	$ \frac{M_L''}{2} $	$ \frac{-2M_L'}{r} $	$ 2M_L'' - \frac{2M_L'}{r} $
$\hat{P}_{lr} r_{mn}$	$ M_L'' $	$ -2M_L'' - \frac{M_L''}{2} r $	$ \frac{M_L''}{2} $	$ \frac{M_L''}{2} $	$ \frac{M_L''}{2} $
$\hat{P}_{mn} r_{lr}$	$ M_L'' $	$ \frac{-2M_L'}{r} $	$ \frac{M_L''}{2} $	$ \frac{M_L''}{2} $	$ 2M_L'' - \frac{2M_L'}{r} $
$\hat{P}_{mr} \hat{P}_{ln}$	$ \frac{M_L'}{r} $	$ \frac{M_L'}{2r} $	$ \frac{-3M_L'}{2r} - \frac{M_L''}{2} $	$ \frac{M_L'}{2r} $	$ \frac{M_L'}{2r} - \frac{M_L''}{2} $
$\hat{P}_{lm} \hat{P}_{rn}$	$ \frac{M_L'}{r} $	$ \frac{M_L'}{2r} $	$ \frac{M_L'}{2r} $	$ \frac{-3M_L'}{2r} - \frac{M_L''}{2} $	$ \frac{M_L'}{2r} - \frac{M_L''}{2} $
$\hat{P}_{mn} \hat{P}_{lr}$	$ \frac{M_L'}{r} $	$ \frac{-3M_L'}{2r} - \frac{M_L''}{2} $	$ \frac{M_L'}{2r} $	$ \frac{M_L'}{2r} $	$ \frac{M_L'}{2r} - \frac{M_L''}{2} $

TABLE 3. The basis tensor components for all fourth order tensors involved in Eq. (3.28). Note that here  $\tilde{T}_{mnrs}$  is as in Eq. (3.23)

Terms	$[r_j r_i M_{jl}]_{,mnrs}$	$-2 [r_l M_{(ml)}]_{,nrs}$	$2M_{mn,rs}$	Sum/ $r^2$
$r_{mnrs}$	$M_L'''' r^2 + 8M_L''' r + 12M_L''$	$-8M_L''' r - 24M_L''$	$12M_L''$	$M_L''''$
$\hat{P}_{ln} r_{mr}, \hat{P}_{lr} r_{nm}, \hat{P}_{lm} r_{rn}$ $\hat{P}_{mr} r_{ln}, \hat{P}_{mn} r_{lr}, \hat{P}_{rn} r_{lm}$	$M_L''' r + 4M_L''$	$-8M_L''$	$M_L''' r + \frac{4M_L'}{r}$	$\frac{2M_L'''}{r} - \frac{4M_L''}{r^2} + \frac{4M_L'}{r^3}$
$\hat{P}_{mr} \hat{P}_{ln}, \hat{P}_{mn} \hat{P}_{lr}, \hat{P}_{rn} \hat{P}_{lm}$	$M_L'' + \frac{3M_L'}{r}$	$-\frac{8M_L'}{r}$	$2M_L'' + \frac{2M_L'}{r}$	$\frac{3M_L''}{r^2} - \frac{3M_L'}{r^3}$